Reconstructing data from alias-free discrete series of samples

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Abstract. There exists a widespread belief that a discrete series of samples does not contain enough information to reconstruct the continuous process that these samples represent. This article demonstrates that for a given series of discrete points there exists only one continuous process that has the bandwidth precisely equal to the Nyquist frequency.

Introduction

When discrete data is "alias-free" it contains discrete samples of a continuous process than had bandwidth limited to *f/2* (the Nyquist frequency), where *f* was the sampling frequency. Reconstruction of the continuous real process from discrete series of samples should respect and preserve the bandwidth *f/2*.

Using piecewise functions such as polynomials (splines) to interpolate between points is very attractive and visually appealing, but unfortunately it ignores the bandwidth of the process that has been sampled. This article presents a Fourier series reconstruction method that guarantees the *f/2* bandwidth.

Fourier series reconstruction

Continuous periodic function $x(t)$ that has limited bandwidth can always be written as a Fourier series with limited number of terms n :

$$x(t) = a_o + 2\sum_{k=1}^{n} \left(a_k \cos\left(2\pi\, kt/T\right) + b_k \sin\left(2\pi\, kt/T\right)\right) \quad (1)$$

where

$$a_k = \frac{1}{T}\int_0^T x(t)\cos\left(2\pi\, kt/T\right)dt \quad \text{and} \quad b_k = \frac{1}{T}\int_0^T x(t)\sin\left(2\pi\, kt/T\right)dt \quad (2)$$

When N discrete samples of $x(t)$ at sampling interval Δ are available in the period T, coefficients a_k and b_k can be efficiently computed using Digital Fourier Transform (or its numerically efficient FFT implementation) of the form

$$X_k = \sum_{r=0}^{N-1} x_r e^{-i\frac{2\pi kr}{N}} \text{, noticing that } X_k = a_k - ib_k \quad . \quad (3)$$

Coefficients X_k are unique up to $k=N/2$. In other words, FFT computes N Fourier series coefficients ($N/2$ a_k coefficients and $N/2$ b_k coefficients) from N discrete data samples. For this reason $n=N/2$ in equation (1).

Since $T=N\Delta$, the highest harmonic ($k=N/2$) in this reconstruction has frequency π/Δ [rad/s], which is exactly half of the sampling frequency $f = 2\pi/\Delta$.

Having coefficients a_k and b_k, the value of continuous function $x(t)$ can be determined for any value of $t \in <0, T>$.

Reconstructed function $x(t)$ passes through each and every point of the discrete series x_k , $k=1...N$ used in equation (3) to determine Fourier series coefficients a_k and b_k.

Computing averages of *x(t)*

Average value $\hat{x}_{t=c}^{t=d}$ of reconstructed function *x(t)* over chosen interval from *t=c* to *t=d* can be computed from analytically determined integral *I(t)* of *x(t)* of the form

$$I(t) = \int x(t)dt = a_o t + \frac{T}{\pi k}\sum_{k=1}^{n}\left(a_k \sin\left(2\pi\, kt/T\right) - b_k \cos\left(2\pi\, kt/T\right)\right) \tag{4}$$

as follows:

$$\hat{x}_{t=c}^{t=d} = \frac{I(d) - I(c)}{d - c} \tag{5}$$

Functions *x(t)* and *I(t)* can be computed together because they share identical values of sin and cos functions (please compare formulae (1) and (4)).

This method of computing averages of *x(t)* is not only accurate but also efficient, because only 2 values *I(c)* and *I(d)* need to be computed. Since *I(t)* is an analytical function, *c* and *d* values can be chosen anywhere within the time interval *T*.

Practical implementation

1. The Fourier series reconstruction presented above is exact only when the sampled and reconstructed function *x(t)* is periodic. In practice it is sufficient that the function *x(t)* begins and ends at the similar value. Choosing a suitable segment *T* from the sampled data can easily satisfy this condition.
2. The theoretical accuracy of coefficients a_k and b_k increases with increasing the number of samples *N* in the time interval *T*. On the other hand numerical errors increase with *N*. For this reason it is expected that for any given precision of computation there will be a practical limit to *N*.
3. For small *N*, Fourier series coefficients can be computed directly from equations (2). For larger *N* more time-efficient is using FFT defined by equation (3).
4. Implementation of the Fourier series reconstruction is very easy. All formulae are simple, deterministic and there are no iterations or ambiguities.

References

1. D.E. Newland, Random Vibrations and Spectral Analysis, Longman, 1975
2. T.J. Chalko, http://sci-e-research.com/Fourier.zip (Software demonstrating the Fourier reconstruction method described in this article)